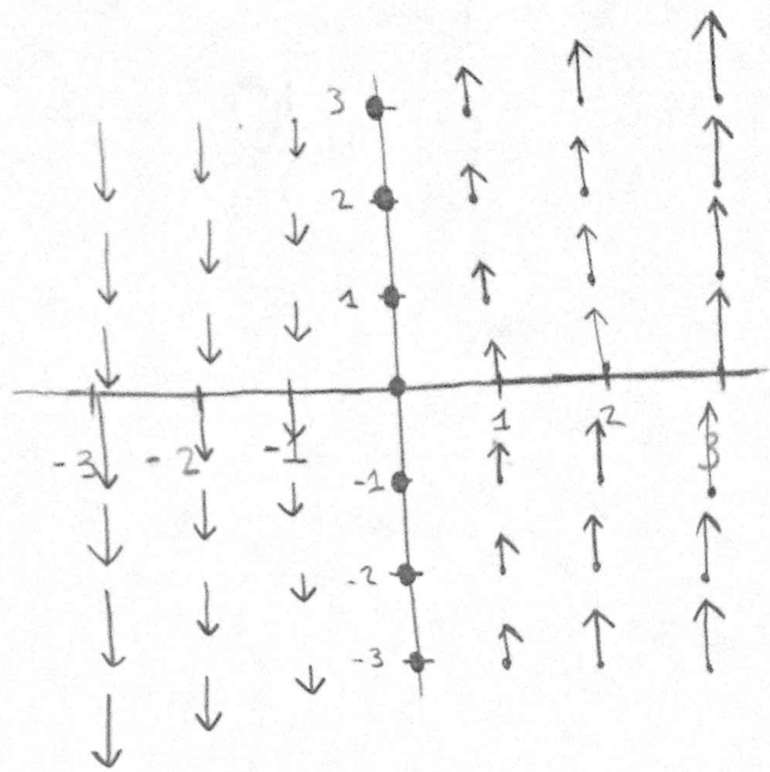
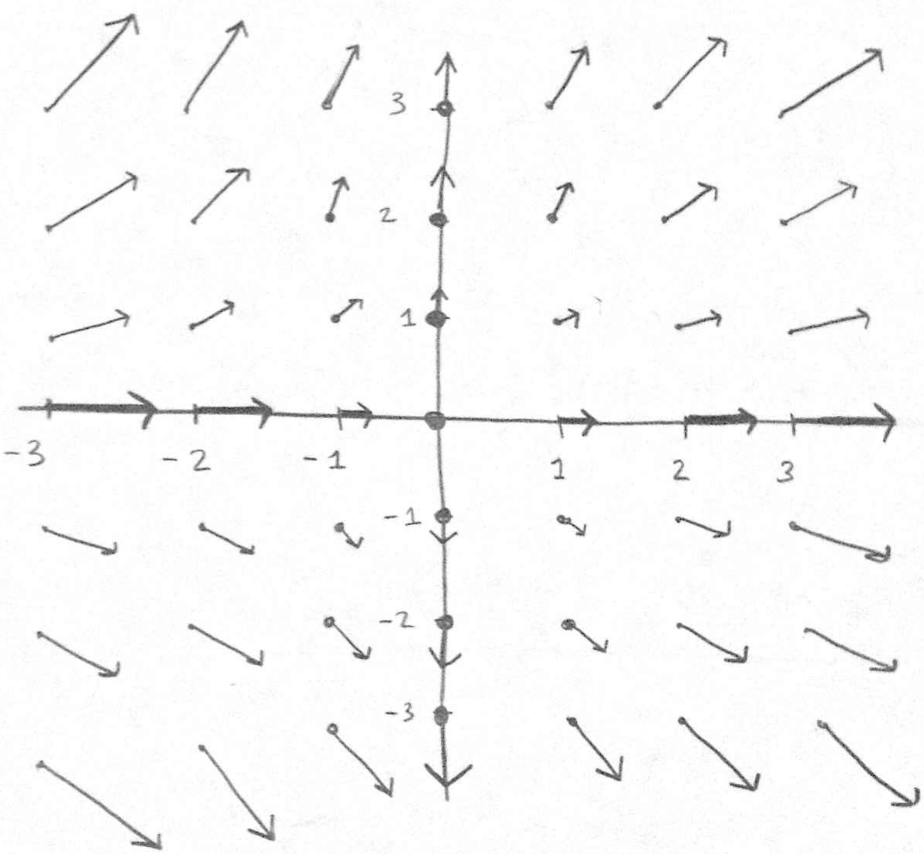
Section 17.1–17.2 WKST SOLUTIONS

1. a) $F = \langle 0, x \rangle$



b) $F = x^2 \mathbf{i} + y \mathbf{j}$

Note: there is symmetry about the x-axis.



2 a. $F = \langle xy, yz, y^2 - x^3 \rangle$

$$\mathrm{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= y + z + 0 = \underline{y + z}.$$

$$\mathrm{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k$$

$$= \langle 2y - y, 3x^2, -x \rangle$$

$$= \underline{\langle y, 3x^2, -x \rangle}.$$

b. $\sin(x+z)\mathbf{i} - ye^{xz}\mathbf{k}$

$$\operatorname{div}(F) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \cos(x+z) + 0 - xye^{xz}$$

$$= \underline{\cos(x+z) - xye^{xz}}.$$

$$\operatorname{curl}(F) = \nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$$= \left\langle -e^{xz} - 0, -\left(-xye^{xz} - \sin(x+z)\right), 0 - 0 \right\rangle$$

$$= \underline{\left\langle -e^{xz}, zye^{xz} + \cos(x+z), 0 \right\rangle}.$$

3. Finding a potential function for $F = \langle x, 0 \rangle$:

We need $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \implies$$

$$f(x,y) = \int x\,dx \quad \text{and} \quad f(x,y) = \int 0\,dy$$

$$= \frac{x^2}{2} + C_1(y) \qquad\qquad = C + C_2(x).$$

So, $\boxed{f(x,y) = \frac{x^2}{2} \text{ works}}$ ↳

method 1: We know that if G is conservative,

then $\text{curl}\, G = 0 \Rightarrow \dfrac{\partial G_1}{\partial y} = \dfrac{\partial G_2}{\partial x}, \dfrac{\partial G_2}{\partial z} = \dfrac{\partial G_3}{\partial y}, \dfrac{\partial G_3}{\partial x} = \dfrac{\partial G_1}{\partial z}.$

But, $\dfrac{\partial G_1}{\partial y} = 1 \neq \dfrac{\partial G_2}{\partial x} = 0.$

So, $G = \langle y, 0 \rangle$ is not conservative.

method 2: A potential function $g(x,y)$ for G would have to satisfy,

$\dfrac{\partial g}{\partial x} = y$ and $\dfrac{\partial g}{\partial y} = 0 \Rightarrow$

$g(x,y) = \int y \, dx$ and $g(x,y) = \int 0 \, dy$

$xy + c(y)$ $= \text{const.} + c(x)$

this condition says $g(x,y)$ is dependent on y.

this condition says that g is not dependent on y.

So, no such g exists. Hence, $G = \langle y, 0 \rangle$ is not conservative.

a. $F = \langle x, y \rangle$ a potential function $f(x,y)$

must satisfy:

$$\frac{\partial f}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = y \Rightarrow$$

$$f(x,y) = \frac{x^2}{2} + C_1(y) \text{ and } f(x,y) = \frac{y^2}{2} + C_2(x).$$

$$\boxed{\text{So, } f(x,y) = \frac{x^2+y^2}{2} \text{ works}}$$

b. $F = \langle yz^2, xz^2, 2xyz \rangle$ a potential function $f(x,y,z)$

must satisfy:

$$\frac{\partial f}{\partial x} = yz^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = xz^2 \quad \text{and} \quad \frac{\partial f}{\partial z} = 2xyz$$

$$\Longrightarrow$$

$$f(x,y,z) = \underline{xyz^2} + C_1(y,z), \quad f(x,y,z) = \underline{xz^2y} + C_2(x,z),$$

$$f(x,y,z) = \underline{xyz^2} + C_3(x,y)$$

So, $\boxed{f(x,y,z) = xyz^2 \text{ works}}$

c. $F = \langle yz\cos(xyz),\ xz\cos(xyz),\ xy\cos(xyz)\rangle$

a potential function $f(x,y,z)$ must satisfy:

$$\frac{\partial f}{\partial x} = yz\cos(xyz), \quad \frac{\partial f}{\partial y} = xz\cos(xyz), \quad \frac{\partial f}{\partial z} = xy\cos(xyz).$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Rightarrow$$

$$f(x,y,z) = \sin(xyz) \qquad f(x,y,z) = \sin(xyz) \qquad f(x,y,z) = \sin(xyz)$$

So, $\boxed{f(x,y,z) = \sin(xyz) \text{ works}}$

Section 17.3 Additional Exercises

1. The Fundamental theorem for conservative Vector Fields: If $F = \nabla f$, then $\displaystyle\int_C F\cdot dr = f(Q) - f(P)$

for any path r from P to Q.

So, $\displaystyle\int_C F\cdot dr = f(1,1,\pi) - f(0,0,0)$

$= 1\cdot 1\sin(\pi) - 0 = 0.$

$F = \langle x^2, y^2 \rangle$. First we note that F is a conservative Force, since a potential function satisfying:

$$\frac{\partial f}{\partial x} = x^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = y^2 \quad \text{is}$$

$$f(x,y) = \frac{x^3}{3} + C_2(y) \quad \text{and} \quad f(x,y) = \frac{y^3}{3} + C_2(x).$$

So, $f(x,y) = \frac{x^3+y^3}{3}$ works.

Recall that work done by the field is $\int_C F \cdot dr$

and work done against the field "i.e. work to ~~move a particle~~" is $-\int_C F \cdot dr$.

So, in our case $W_{total} = W_{0 \text{ to } Q} = -f(Q) + f(0)$

$\rightarrow \boxed{\frac{-2}{3}}$

neg. sign makes sense since force is in the direction of motion.

Since the force is conservative the work moving in a complete circuit around the square is 0.

3. Evaluate $\oint_C \sin(x)\,dx + z\cos(y)\,dy + \sin y\,dz$

where C is the ellipse $4x^2+9y^2=36$.

$F = \langle \sin x,\ z\cos y,\ \sin y\rangle$. If $\nabla f = F$ then,

$\frac{\partial f}{\partial x} = \sin x$, $\frac{\partial f}{\partial y} = z\cos y$, $\frac{\partial f}{\partial z} = \sin y \Rightarrow$

$f(x,y,z) = -\cos(x) + C_1(y,z)$

$f(x,y,z) = z\sin y + C_2(x,z)$

$f(x,y,z) = z\sin y + C_3(x,y)$

So, $f(x,y,z) = z\sin y - \cos(x)$ is a potential function hence the Force is conservative.

Since C is a closed curve,

$$\boxed{\oint_C \sin(x)\,dx + z\cos(y)\,dy + \sin(y)\,dz = 0}$$

Additional Exercises 17.2

1. (a) The path C_1 from $(0,0,1)$ to $(0,2,0)$ has parametrization $r_1(t) = \langle 0,0,1\rangle + t\langle 0,2,-1\rangle$ $0 \le t \le 1$ and the path, C_2, from $(0,2,0)$ to $(1,1,1)$ has parametrization $r_2(t) = \langle 0,2,0\rangle + t\langle 1,-1,1\rangle$ $0 \le t \le 1$.

Moreover, we have $r_1'(t) = \langle 0,2,-1\rangle \Rightarrow$

$\|r_1'(t)\| = \sqrt{0^2+2^2+(-1)^2} = \sqrt{5}$ and

$r_2'(t) = \langle 1,-1,1\rangle \Rightarrow \|r_2'(t)\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$.

$f(r_1(t)) = 0$ and $f(r_2(t)) = te^{t^2}$.

So, $$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \int_0^1 0\cdot\sqrt{5}\,dt + \sqrt{3}\int_0^1 te^{t^2}\,dt$$

$u = t^2$
$du = 2t\,dt$

$$= \frac{\sqrt{3}}{2}\int_0^1 e^u\,du = \boxed{\sqrt{3}/2\,(e-1)}$$

(b) We can parametrize the curve by sending $x \to t$ and it becomes $r(t) = \langle t, t^3 \rangle \quad 0 \le t \le 1$.

Then, $r'(t) = \langle 1, 3t^2 \rangle \Rightarrow \|r'(t)\| = \sqrt{1 + 9t^4}$.

$f(r(t)) = \sqrt{1 + 9t \cdot t^3} = \sqrt{1+9t^4}$.

So, $$\int_C f\,ds = \int_0^1 f(r(t))\|r'(t)\|dt = \int_0^1 1 + 9t^4 \, dt$$

$$= t\Big|_0^1 + \frac{9t^5}{5}\Big|_0^1 = 1 + 9/5 = \boxed{14/5}$$

2. (a) First we parametrize the segment from $(0,0)$ to $(2,2)$

$$r(t) = t\langle 2,2 \rangle = \langle 2t, 2t \rangle, \quad 0 \le t \le 1$$

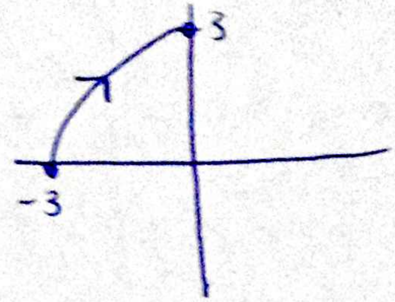
Then, $r'(t) = \langle 2,2 \rangle$, and $F(r(t)) = \langle (2t)^2, (2t)(2t) \rangle$
$= \langle 4t^2, 4t^2 \rangle$.

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\,dt = \int_0^1 8t^2 + 8t^2 dt = 16\int_0^1 t^2 dt$$

$$= \boxed{\frac{16}{3}}$$

2 (b) A parametrization for the curve is

$r(t) = \langle 3\cos t, -3\sin t \rangle \quad \pi \le t \le 3\pi/2$



Then, $r'(t) = \langle -3\sin t, -3\cos t \rangle$ and

$F(r(t)) = \langle 9\cos^2 t, -9\cos t \sin t \rangle$.

So, $F(r(t)) \cdot r'(t) = -27\cos^2 t \sin t + 27\cos^2 t \sin t$

$= 0.$

Hence $\displaystyle\int_C F \cdot dr = \int_{\pi}^{3\pi/2} F(r(t)) \cdot r'(t)\, dt = \boxed{0}$

3. We can parametrize the parabola by

$r(t) = \langle t, t^2 \rangle$ for $0 \le t \le 2.$

Then, $r'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \langle 1, 2t \rangle$

So, $y\,dx - x\,dy = (y - x(2t))\,dt$

$= t^2 - 2t^2\, dt.$

$= -t^2 dt$

So $\displaystyle\int_C y\,dx - x\,dy = -\int_0^2 t^2\, dt = -\frac{t^3}{3}\Big|_0^2 = \boxed{-8/3}$